

# Chromatic number of ordered graphs with forbidden ordered subgraphs.

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## Abstract

It is well-known that the graphs not containing a given graph  $H$  as a subgraph have bounded chromatic number if and only if  $H$  is acyclic. Here we consider *ordered graphs*, i.e., graphs with a linear ordering  $\prec$  on their vertex set, and the function

$$f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\},$$

where  $\text{Forb}_{\prec}(H)$  denotes the set of all ordered graphs that do not contain a copy of  $H$ .

If  $H$  contains a cycle, then as in the case of unordered graphs,  $f_{\prec}(H) = \infty$ . However, in contrast to the unordered graphs, we describe an infinite family of ordered forests  $H$  with  $f_{\prec}(H) = \infty$ . An ordered graph is crossing if there are two edges  $uv$  and  $u'v'$  with  $u \prec u' \prec v \prec v'$ . For connected crossing ordered graphs  $H$  we reduce the problem of determining whether  $f_{\prec}(H) \neq \infty$  to a family of so-called monotonically alternating trees. For non-crossing  $H$  we prove that  $f_{\prec}(H) \neq \infty$  if and only if  $H$  is acyclic and does not contain a copy of any of the five special ordered forests on four or five vertices, which we call bonnets. For such forests  $H$ , we show that  $f_{\prec}(H) \leq 2^{|V(H)|}$  and that  $f_{\prec}(H) \leq 2|V(H)| - 3$  if  $H$  is connected.

**Keywords:** ordered graphs, chromatic number, forbidden subgraphs

## 1 Introduction

What conclusions can one make about the chromatic number of a graph knowing that it does not contain certain subgraphs? Let  $H$  be a graph on at least two vertices,  $\text{Forb}(H)$  be the set of all graphs not containing  $H$  as a subgraph, and  $f(H) = \sup\{\chi(G) \mid G \in \text{Forb}(H)\}$ . If  $H$  has a cycle of length  $\ell$ , then for any integer  $\chi$  there is a graph  $G$  of girth at least  $\ell+1$  and chromatic number  $\chi$ , see [11], implying that  $f(H) = \infty$ . On the other hand, if  $H$  is a forest on  $k$  vertices and  $G$  is a graph of chromatic number at least  $k$ , then  $G$  contains a  $k$ -critical subgraph  $G'$ , that in turn

has minimum degree at least  $k - 1$ . Thus a copy of  $H$  can be found as a subgraph of  $G'$  by a greedy embedding. Therefore  $G \notin \text{Forb}(H)$ , implying that  $f(H) \leq k - 1$ . So, we see that  $f(H)$  is finite if and only if  $H$  is acyclic.

A similar situation holds for directed graphs, with a similarly defined function  $f_{\text{dir}}(H)$  being finite if and only if the underlying graph of  $H$  is acyclic. A result of Addalirio-Berry *et al.* [1], see also [4], implies that  $f_{\text{dir}}(H) \leq k^2/2 - k/2 - 1$  whenever  $H$  is a directed  $k$ -vertex graph whose underlying graph is acyclic.

Here, we consider the behavior of the chromatic number of ordered graphs with forbidden ordered subgraphs. An *ordered graph*  $G$  is a graph  $(V, E)$  together with a linear ordering  $\prec$  of its vertex set  $V$ . An *ordered subgraph*  $H$  of an ordered graph  $G$  is a subgraph of the (unordered) graph  $(V, E)$  together with the linear ordering of its vertices inherited from  $G$ . An ordered subgraph  $H$  is a *copy of an ordered graph*  $H'$  if there is an order preserving isomorphism between  $H$  and  $H'$ . For an ordered graph  $H$  on at least two vertices<sup>1</sup> let  $\text{Forb}_{\prec}(H)$  denote the set of all ordered graphs that do not contain a copy of  $H$ . We consider the function  $f_{\prec}$  given by

$$f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\}.$$

We show that it is no longer true that  $f_{\prec}(H)$  is finite if and only if  $H$  is acyclic. When  $H$  is connected, we reduce the problem of determining whether  $f_{\prec}(H) \neq \infty$  to a well behaved class of trees, which we call monotonically alternating trees. We completely classify so-called “non-crossing” ordered graphs  $H$  for which  $f_{\prec}(H) = \infty$ . In case of “non-crossing”  $H$  with finite  $f_{\prec}(H)$ , we provide specific upper bounds on this function in terms of the number of vertices in  $H$ . Note that  $f_{\prec}(H) \geq |V(H)| - 1$  for any ordered graph  $H$ , since a complete graph on  $|V(H)| - 1$  vertices is in  $\text{Forb}_{\prec}(H)$ .

We need some formal definitions before stating the main results of the paper. We consider the vertices of an ordered graph laid out along a horizontal line according to their ordering  $\prec$  and say that for  $u \prec v$  the vertex  $u$  is to the *left of*  $v$  and the vertex  $v$  is to the *right of*  $u$ . We write  $u \preceq v$  if  $u \prec v$  or  $u = v$ . For two sets of vertices  $U$  and  $U'$  we write  $U \prec U'$  if all vertices in  $U$  are left of all vertices in  $U'$ . Two edges  $uv$  and  $u'v'$  **cross** if  $u \prec u' \prec v \prec v'$  and an ordered graph  $H$  is called *crossing* if it contains two crossing edges. Otherwise,  $H$  is called *non-crossing*. Two distinct ordered graphs  $G$  and  $H$  *cross each other* if there is an edge in  $G$  crossing an edge in  $H$ .

An ordered graph is a **bonnet** if it has 4 or 5 vertices  $u_1 \prec u_2 \preceq u_3 \prec u_4 \preceq u_5$  and edges  $u_1u_2, u_1u_5, u_3u_4$ , or if it has vertices  $u_1 \preceq u_2 \prec u_3 \preceq u_4 \prec u_5$  and edges  $u_1u_5, u_4u_5, u_2u_3$ . See Figure 1 (first two rows). An ordered path  $P = u_1, \dots, u_n$  is a **tangled path** if for a vertex  $u_i$ ,  $1 < i < n$ , that is either leftmost or rightmost in  $P$  there is an edge in the subpath  $u_1, \dots, u_i$  that crosses an edge in the subpath

<sup>1</sup>If  $H$  has only one vertex, then  $\text{Forb}_{\prec}(H)$  consists only of the graph with empty vertex set and one can think of  $f_{\prec}(H)$  as being equal to 0. However, we will avoid this pathologic case throughout.

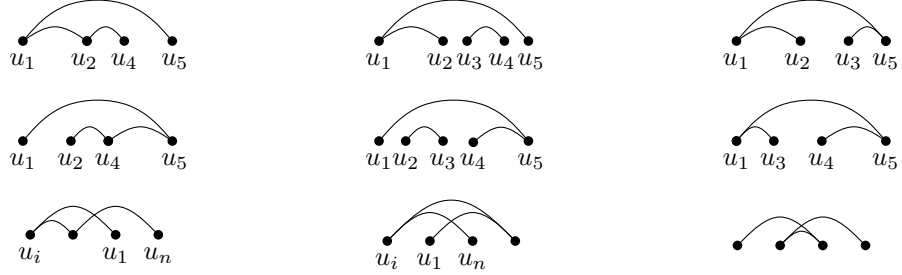


Figure 1: All bonnets (first two rows), two tangled paths (last row, left and middle) and a crossing path that is not tangled (last row, right).

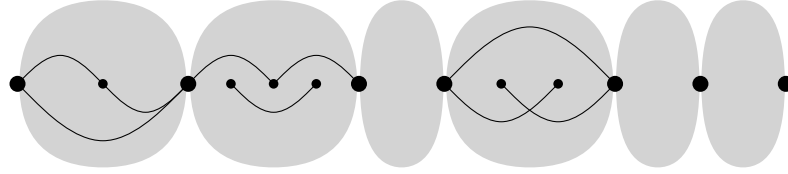


Figure 2: Segments of an ordered graph. The bold vertices are either inner cut-vertices or left-, rightmost vertices.

$u_i, \dots, u_n$ . See Figure 1 (last row, left and middle). Note that there are crossing paths which are not tangled, see for example Figure 1 (right).

**Theorem 1.** *If an ordered graph  $H$  contains a cycle, a bonnet, or a tangled path, then  $f_{\prec}(H) = \infty$ .*

A vertex  $v$  of an ordered graph  $G$  is called *inner cut vertex*, if there is no edge  $uw$  with  $u \prec v \prec w$  in  $G$  and  $v$  is not leftmost or rightmost in  $G$ . An *interval* in an ordered graph  $G$  is a set  $I$  of vertices such that for all vertices  $u, v \in I$ ,  $x \in V(G)$  with  $u \prec x \prec v$  we have  $x \in I$ . A **segment** of an ordered graph  $G$  with  $|V(G)| \geq 2$  is an induced subgraph  $H$  of  $G$  such that  $|V(H)| \geq 2$ ,  $V(H)$  is an interval in  $G$ , the leftmost and rightmost vertices in  $H$  are either inner cut vertices of  $G$  or leftmost respectively rightmost in  $G$ , and all other vertices in  $H$  are not inner cut vertices in  $G$ . So,  $G$  is the union of its segments, any two segments share at most one vertex and the inner cut vertices of  $G$  are precisely the vertices contained in two segments of  $G$ . In particular, the number of inner cut vertices of  $G$  is exactly one less than the number of its segments. See Figure 2.

The *length* of an edge  $xy$  is the number of vertices  $v$  such that  $x \preceq v \prec y$ . A shortest edge among all the edges incident to a vertex  $x$  is referred to as a *shortest edge incident to  $x$* . Note that there is either 1 or 2 shortest edges incident to a given vertex in a connected graph on at least two vertices. Let  $U$  be a vertex set in an ordered tree  $T$ , such that each vertex in  $U$  has exactly one shortest edge incident to it. For such a set  $U$ , let  $S(U)$  be the set of edges  $e_u$  such that  $e_u$  is a shortest edge incident to  $u$ ,  $u \in U$ . We call an ordered tree  $T$  **monotonically alternating** if there is a partition  $V(T) = L \dot{\cup} R$ , with  $L \prec R$ , such that  $L$  and  $R$  are independent sets in  $T$ ,  $E = S(L) \cup S(R)$ , and neither  $S(L)$  nor  $S(R)$  contains a pair of crossing

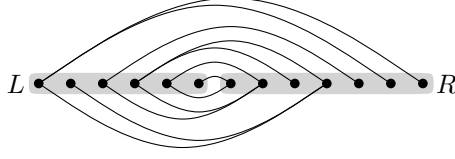


Figure 3: A monotonically alternating tree. Each edge on top is the shortest edge incident to a vertex in  $R$  and each edge at the bottom is the shortest edge incident to a vertex in  $L$ .

edges.

**Theorem 2.** *An ordered tree  $T$  contains neither a bonnet nor a tangled path if and only if each segment of  $T$  is monotonically alternating. In particular if  $f_{\prec}(H) \neq \infty$  for some connected ordered graph  $H$ , then each segment in  $H$  is a monotonically alternating tree.*

Recall that an ordered graph is non-crossing if it does not contain any crossing edges. Note that a non-crossing graph does not contain tangled paths.

**Theorem 3.** *Let  $T$  be a non-crossing ordered graph on  $k$  vertices. Then  $f_{\prec}(T) \neq \infty$  if and only if  $T$  is a forest that does not contain a bonnet.*

*Moreover, if  $f_{\prec}(T) \neq \infty$  then  $k-1 \leq f_{\prec}(T) \leq 2^k$ . If, in addition  $T$  is connected, then  $f_{\prec}(T) \leq 2k-3$ . Finally, for each  $k \geq 4$  there is an ordered non-crossing tree  $T$  with  $k \leq f_{\prec}(T) \neq \infty$ , while for  $k=2,3$  we have  $f_{\prec}(T) = k-1$ .*

For certain classes of ordered forests we prove better upper bounds on  $f_{\prec}$ . A  $k$ -nesting is an ordered graph  $T$  on vertices  $u_1 \prec \dots \prec u_k \prec v_k \prec \dots \prec v_1$  and edges  $u_i v_i$ ,  $1 \leq i \leq k$ . A  $k$ -crossing is an ordered graph  $T$  on vertices  $u_1 \prec \dots \prec u_k \prec v_1 \prec \dots \prec v_k$  and edges  $u_i v_i$ ,  $1 \leq i \leq k$ . We may omit the parameter  $k$  if it is not important. A *generalized star* is a union of a star and isolated vertices.

The following theorem summarizes several results on trees which are either not covered by Theorem 3 or improve the upper bound from Theorem 3 significantly.

One of the known classes of such graphs is a special family of star forests, or, in other words, tuple matchings. For positive integers  $m$  and  $t$  and a permutation  $\pi$  of  $[t]$ , an  $m$ -tuple  $t$ -matching  $M = M(t, m, \pi)$  is an ordered graph with vertices  $v_1 \prec \dots \prec v_{t(m+1)}$ , where each edge is of the form  $v_i v_{t+j+m(\pi(i)-1)}$  for  $1 \leq i \leq t$ ,  $1 \leq j \leq m$ . I.e., an  $m$ -tuple  $t$ -matching is a vertex disjoint union of  $t$  stars on  $m$  edges each, where  $v_1, \dots, v_t$  are the centers of the stars that are to the left of all leaves and the leaves of each star form an interval in  $M$ , so that these intervals are ordered according to the permutation  $\pi$ . The third item in the following theorem is an immediate corollary of a result by Weidert [19] who provides a linear upper bound on the extremal function for  $M$ . The other results are based on linear upper bounds for the extremal functions of nestings due to Dujmovic and Wood [10], on the extremal function of crossings due to Capovileas and Pach [5] and lower bounds for ordered Ramsey numbers due to Conlon *et al.* [7], see also Balko *et al.* [2]. See Section 3 for a more detailed description of extremal functions and ordered Ramsey numbers.

**Theorem 4.** *Let  $T$  be an ordered forest on  $k$  vertices.*

- *If each segment of  $T$  is either a generalized star, a 2-nesting, or a 2-crossing, then  $f_{\prec}(T) = k - 1$ .*
- *If each segment of  $T$  is either a nesting, a crossing, a generalized star, or a non-crossing tree without bonnets, then  $k - 1 \leq f_{\prec}(T) \leq 2k - 3$ .*
- *If  $T$  is a tuple matching, then  $k - 1 \leq f_{\prec}(T) \leq 2^{10k \log(k)}$ .*
- *There is a positive constant  $c$  such that for each even positive integer  $k \geq 4$  there is a matching  $M$  on  $k$  vertices with  $f_{\prec}(M) \geq 2^{c \frac{\log(k)^2}{\log \log(k)}}$ .*

The paper is organized as follows. In Section 2 we introduce all missing necessary notions. In Section 3 we summarize the known results on extremal functions and Ramsey numbers for ordered graphs and show how they could be used in determining  $f_{\prec}$ . In Section 4 we prove some structural lemmas and provide several reductions that are used in the proofs of the main results and that might be of independent interest. Section 5 contains the proofs of Theorems 1–4. We summarize all known results for forests with at most three edges in Section 6. Finally, Section 7 contains conclusions and open questions.

## 2 Definitions

Let  $K_n$  denote a complete graph on  $n$  vertices. For a positive integer  $n$  and an ordered graph  $H$ , let  $\text{ex}_{\prec}(n, H)$  denote the *ordered extremal number*, i.e., the largest number of edges in an ordered graph on  $n$  vertices in  $\text{Forb}_{\prec}(H)$ . For an ordered graph  $H$  the *ordered Ramsey number*  $R_{\prec}(H)$  is the smallest integer  $n$  such that in any edge-coloring of an ordered  $K_n$  in two colors there is a monochromatic copy of  $H$ . Recall that an interval in an ordered graph  $G$  is a set  $I$  of vertices such that for all vertices  $u, v \in I$ ,  $x \in V(G)$  with  $u \prec x \prec v$  we have  $x \in I$ . The *interval chromatic number*  $\chi_{\prec}(G)$  of an ordered graph  $G$  is the smallest number of intervals, each inducing an independent set in  $G$ , needed to partition  $V(G)$ . An inner cut vertex  $v$  of an ordered graph  $G$  *splits*  $G$  into ordered graphs  $G_1$  and  $G_2$  if  $G_1$  is induced by all vertices  $u$  with  $u \preceq v$  in  $G$  and  $G_2$  is induced by all vertices  $u$  with  $v \preceq u$ . A vertex of degree 1 is called a *leaf*. A vertex in an ordered graph  $G$  is called *reducible*, if it is a leaf in  $G$ , is leftmost or rightmost in  $G$  and has a common neighbor with the vertex next to it. We call an edge  $uv$  in a graph  $G$  *isolated* if  $u$  and  $v$  are leaves in  $G$ . A graph  $G$  is  *$t$ -degenerate* if each subgraph of  $G$  has a vertex of degree at most  $t$ . A vertex  $v$  is *between* vertices  $u$  and  $w$  if  $u \preceq v \preceq w$ . The *reverse*  $\overline{G}$  of an ordered graph  $G$  is the ordered graph obtained by reversing the ordering of the vertices in  $G$ . A  *$u$ - $v$ -path*  $P$  is a path starting with  $u$  and ending with  $v$ , i.e., a path  $v_1, \dots, v_k$  with  $u = v_1$ ,  $v = v_k$ . Given a path  $P = v_1, \dots, v_k$  let  $v_i P = v_i, \dots, v_k$  and  $P v_i = v_1, \dots, v_i$ . Similarly for a neighbor  $v \notin V(P)$  of  $v_1$  let  $vP = v, v_1, \dots, v_k$ . If  $U \subseteq V(G)$ ,  $F \subseteq E(G)$  let  $G[U]$ ,  $G - U$  and  $G - F$  denote

the graphs  $(U, E(G) \cap \binom{U}{2})$ ,  $(V(G) \setminus U, E(G) \cap \binom{V(G) \setminus U}{2})$ , and  $(V(G), E(G) \setminus F)$ , respectively. In particular if  $u, v \in V(G)$  then  $G - \{u, v\}$  is the graph obtained by removing  $u$  and  $v$  from  $G$ , not the edge  $uv$  only. If  $u \in V(G)$  let  $G - u = G - \{u\}$ . The definitions of tangled paths, bonnets, crossing edges and subgraphs, intervals, segments, inner cut-vertices, and monotonically alternating trees are given before the statements of the main theorems in the introduction. We shall typically denote a general ordered graph by  $H$ , a tree or a forest by  $T$ , and a larger ordered graph by  $G$ . For all other undefined graph theoretic notions we refer the reader to West [20].

### 3 Connections to known results

There are connections between the extremal number  $\text{ex}_{\prec}(n, H)$  and the function  $f_{\prec}(H)$ . If there is a constant  $c$  such that  $\text{ex}_{\prec}(n, H) < cn$  for every  $n$ , then

$$f_{\prec}(H) \leq 2c, \quad (1)$$

so  $f_{\prec}(H)$  is finite. Indeed, if  $\text{ex}_{\prec}(n, H) < cn$  then any  $G \in \text{Forb}_{\prec}(H)$  has less than  $c|V(G)|$  edges, and hence has a vertex of degree less than  $2c$ . Thus if  $G \in \text{Forb}_{\prec}(H)$ , then each subgraph of  $G$  is in  $\text{Forb}_{\prec}(H)$ , so each subgraph has a vertex of degree less than  $2c$ , so  $G$  is  $(2c - 1)$ -degenerate. Therefore  $\chi(G) \leq 2c$ .

Ordered extremal numbers are studied in detail in [17]. Recall that  $\chi_{\prec}(G)$  is the smallest number of intervals, each inducing an independent set, needed to partition the vertices of an ordered graph  $G$ . Pach and Tardos [17] prove that for each ordered graph  $H$

$$\text{ex}_{\prec}(n, H) = \left(1 - \frac{1}{\chi_{\prec}(H) - 1}\right) \binom{n}{2} + o(n^2).$$

For ordered graphs with interval chromatic number 2, Pach and Tardos find a tight relation between the ordered extremal number and pattern avoiding matrices. For an ordered graph  $H$  with  $\chi_{\prec}(H) = 2$  let  $A(H)$  denote the 0-1-matrix where the rows correspond to the vertices in the first color and the columns to the vertices in the second color of a proper interval coloring of  $H$  in 2 colors and let  $A(H)_{u,v} = 1$  if and only if  $uv$  is an edge in  $H$ . A 0-1-matrix  $B$  *avoids* another 0-1-matrix  $A$  if there is no submatrix in  $B$  which becomes equal to  $A$  after replacing some ones with zeros. For a 0-1-matrix  $A$  let  $\text{ex}(n, A)$  denote the largest number of ones in an  $n \times n$  matrix avoiding  $A$ . In [17] it is shown that for each ordered graph  $H$  with  $\chi_{\prec}(H) = 2$  there is a constant  $c$  such that  $\text{ex}(\lfloor \frac{n}{2} \rfloor, A(H)) \leq \text{ex}_{\prec}(n, H) \leq c \text{ex}(n, A(H)) \log n$ . Thus, when  $\text{ex}(n, A(H))$  is linear in  $n$ , one can guarantee that  $\text{ex}_{\prec}(n, H) = O(n \log n)$ , but this is not enough to claim that  $f_{\prec}(H) \neq \infty$ .

In addition, we see that there is no direct connection between  $f_{\prec}(H)$  and  $\text{ex}_{\prec}(n, H)$  because there are dense ordered graphs avoiding  $H$  for some ordered graphs  $H$  with small  $f_{\prec}(H)$ . A specific example for such a graph  $H$  is an ordered path  $u_1 u_2 u_3 u_4$ , with  $u_1 \prec u_2 \prec u_3 \prec u_4$ . One can see from Theorem 4 that  $f_{\prec}(H) = 3$ , but a complete bipartite ordered graph  $G$  with all vertices of one bipartition class to the left of all other vertices does not contain  $H$  and has  $|V(G)|^2/4$  edges. However,

for some ordered graphs  $H$  with interval chromatic number 2, one can show that  $\text{ex}_{\prec}(n, H)$  is linear. This in turn, implies that  $f_{\prec}(H)$  is finite.

Some of the extensive research on forbidden binary matrices and extremal functions for ordered graphs can be found in [3, 12, 14, 15, 16].

There are also connections between the Ramsey numbers  $R_{\prec}(H)$  for ordered graphs and the function  $f_{\prec}(H)$ . If the edges of  $K_n$ ,  $n = R_{\prec}(H) - 1$ , are colored in two colors without monochromatic copies of  $H$ , then both color classes form ordered graphs  $G_1$  and  $G_2$  not containing  $H$  as an ordered subgraph. Then one of the  $G_i$ 's has chromatic number at least  $\sqrt{n}$ , since a product of proper colorings of  $G_1$  and  $G_2$  yields a proper coloring of  $K_n$ . Therefore  $f_{\prec}(H) \geq \sqrt{R_{\prec}(H) - 1}$ . Ordered Ramsey numbers were recently studied by Conlon *et al.* [7] and Balko *et al.* [2]. Other research on ordered graphs includes characterizations of classes of graphs by forbidden ordered subgraphs [8, 13] and the study of perfectly ordered graphs [6].

## 4 Structural Lemmas and Reductions

In this section we first analyze the structure of ordered trees without bonnets and tangled paths. This leads to a proof of Theorem 2 in Section 5. Afterwards we establish several cases when  $f_{\prec}(H)$  can be upper bounded in terms of  $f_{\prec}(H')$  for a subgraph  $H'$  of  $H$ . This allows us to reduce the problem of whether  $f_{\prec}(H) \neq \infty$  to the problem of whether  $f_{\prec}(H') \neq \infty$ . These *reductions* are the crucial tools in the proof of Theorem 3 in Section 5.

**Lemma 4.1.** *Let  $T$  be an ordered tree that does not contain a tangled path and let  $u \prec v \prec w$  be vertices in  $T$ . If  $uw$  is an edge in  $T$ , then all vertices of the path connecting  $u$  and  $v$  in  $T$  are between  $u$  and  $w$ .*

*Proof.* Let  $P$  be the path in  $T$  that starts with  $v$  and ends with the edge  $uw$ . Let  $\ell$  denote the leftmost vertex in  $P$ . Assume for the sake of contradiction that  $\ell \prec u$ . Then the path  $vP\ell$  contains neither  $u$  nor  $w$  and therefore crosses the edge  $uw$ . Hence the paths  $P\ell$  and  $\ell P$  cross and  $P$  is tangled, a contradiction. Therefore  $\ell = u$ . Due to symmetric arguments  $w$  is the rightmost vertex in  $P$ . Hence all vertices in  $P$  are between  $u$  and  $w$ .  $\square$

**Lemma 4.2.** *Let  $T$  be an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment. Deleting any leaf from  $T$  yields an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment.*

*Proof.* Let  $uv$  be an edge in  $T$  incident to a leaf  $u$  and let  $T' = T - u$ . Then clearly  $T'$  is an ordered tree that contains neither a bonnet nor a tangled path. For the sake of contradiction assume that  $T'$  has at least two segments and let  $x$  be an inner cut vertex in  $T'$ . Then  $x \neq u, v$  and is between  $u$  and  $v$  in  $T$ , since  $x$  is not an inner cut vertex in  $T$ . By reversing  $T$  if necessary we may assume that  $v \prec x \prec u$ . Let  $P$  be the  $v$ - $x$ -path in  $T'$ . All vertices in  $P$  are between  $v$  and  $u$  by Lemma 4.1 applied to  $u, v$  and  $x$ . In addition no vertex in  $P$  is to the right of  $x$  since  $x$  is an inner cut

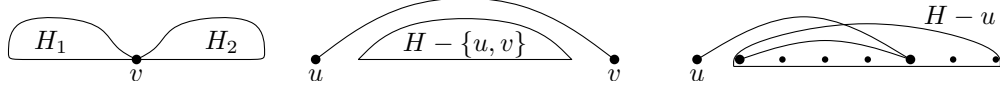


Figure 4: An inner cut vertex  $v$  splitting an ordered graph into ordered graphs  $H_1$  and  $H_2$  (left), an isolated edge  $uv$  in an ordered graph  $H$  (middle), and a reducible vertex  $u$  (right).

vertex in  $T'$ . So all vertices in  $P$  are between  $v$  and  $x$ . Let  $vw$  denote the first edge of  $P$  and let  $xy$  denote an edge in  $T'$  with  $x \prec y$ . Such an edge  $xy$  exists since the inner cut vertex  $x$  is not rightmost in  $T'$  and  $T'$  is connected. If  $u \prec y$ , then  $uvPxy$  is a tangled path in  $T$ . If  $y \prec u$ , then  $u, v, w, x$  and  $y$  form a bonnet in  $T$ . In both cases we have a contradiction and hence  $T'$  has only one segment.  $\square$

**Lemma 4.3.** *If  $T$  is an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment, then  $\chi_{\prec}(T) \leq 2$ .*

*Proof.* We prove the claim by induction on  $k = |V(T)|$ . If  $k \leq 2$ , then clearly  $\chi_{\prec}(T) \leq 2$ . So assume that  $k \geq 3$ . Let  $u$  denote a leaf in  $T$ ,  $v$  its neighbor in  $T$ , and let  $T' = T - u$ . Then  $T'$  has only one segment and contains neither a bonnet nor a tangled path due to Lemma 4.2. Inductively  $\chi_{\prec}(T') \leq 2$ , i.e., there is a partition  $L \dot{\cup} R = V(T')$ , with  $L \prec R$ , such that all edges in  $T'$  are between  $L$  and  $R$ . By reversing  $T$  if necessary we assume that  $v \in L$ . For the sake of contradiction assume that  $\chi_{\prec}(T) > 2$ . Then  $u \prec \ell$  for the rightmost vertex  $\ell$  in  $L$ , possibly  $\ell = v$ . Let  $w \in R$  denote one fixed neighbor of  $v$  in  $T'$ . Then all vertices of the path connecting  $\ell$  and  $v$  in  $T'$  are between  $v$  and  $w$  due to Lemma 4.1. In particular  $\ell$  is incident to an edge  $\ell x$ ,  $x \in R$ , with  $x \preceq w$ . Hence  $u \prec v$ , since otherwise there is a bonnet on vertices  $v, u, \ell, x$ , and  $w$  in  $T$ . If there is a vertex  $y$ ,  $u \prec y \prec v$ , then all vertices of the path connecting  $y$  and  $u$  in  $T$  are between  $u$  and  $v$  due to Lemma 4.1. But this is not possible since  $y, v \in L$  and all the neighbors of  $y$  are in  $R$ . Hence  $u$  is immediately to the left of  $v$  in  $T$ . Note that  $u$  is not leftmost in  $T$ , since otherwise  $v$  is an inner cut vertex in  $T$ . Consider the path  $P$  connecting a vertex left of  $u$  to  $\ell$  in  $T$ . This path contains distinct vertices  $p, q \in L$ ,  $r \in R$ , such that  $pr$  and  $rq$  are edges in  $P$  and  $p \prec u \prec v \preceq q \prec r$ . Hence there is a bonnet, a contradiction. This shows that  $\chi_{\prec}(T) \leq 2$ .  $\square$

We now present several reductions. Let us mention that some of the following arguments are similar to reductions used for extremal numbers of matrices [17, 18].

Recall, that an inner cut vertex  $v$  of an ordered graph  $H$  splits  $H$  into ordered graphs  $H_1$  and  $H_2$ , where  $H_1$  is induced by all vertices  $u$  with  $u \preceq v$  in  $H$  and  $H_2$  is induced by all vertices  $u$  with  $v \preceq u$ . See Figure 4 (left).

**Reduction Lemma 1.** *If an inner cut vertex  $v$  splits an ordered graph  $H$  into ordered graphs  $H_1$  and  $H_2$  with  $f_{\prec}(H_1), f_{\prec}(H_2) \neq \infty$ , then*

$$f_{\prec}(H) \leq f_{\prec}(H_1) + f_{\prec}(H_2).$$



*Proof.* Consider an ordered graph  $G \in \text{Forb}_{\prec}(H)$ . Let  $V_1$  denote the set of vertices in  $G$  that are rightmost in some copy of  $H_1$  in  $G$ . Further let  $V_2 = V(G) \setminus V_1$ . Then  $G[V_2] \in \text{Forb}_{\prec}(H_1)$  by the choice of  $V_1$ . Moreover  $G[V_1] \in \text{Forb}_{\prec}(H_2)$ , since otherwise the leftmost vertex  $u$  in a copy of  $H_2$  in  $G[V_1]$  is also a rightmost vertex in a copy of  $H_1$  and hence plays the role of  $v$  in a copy of  $H$  in  $G$ . Thus  $\chi(G) \leq \chi(G[V_1]) + \chi(G[V_2]) \leq f_{\prec}(H_2) + f_{\prec}(H_1)$  and since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq f_{\prec}(H_1) + f_{\prec}(H_2)$ .  $\square$

**Reduction Lemma 2.** *If  $v$  is an isolated vertex in an ordered graph  $H$  with  $|V(H)| \geq 3$  and  $f_{\prec}(H - v) \neq \infty$ , then  $f_{\prec}(H) \leq 2f_{\prec}(H - v)$ .*

*Proof.* Consider an ordered graph  $G \in \text{Forb}_{\prec}(H)$ . If  $v$  is not leftmost or rightmost in  $H$ , then let  $V_1$  denote a set of every other vertex in  $G$  and let  $V_2 = V(G) \setminus V_1$ . Then  $G[V_1], G[V_2] \in \text{Forb}_{\prec}(H - v)$ , since for any two vertices  $u \prec w$  in  $V_i$  there is a vertex  $v \in V_{3-i}$  with  $u \prec v \prec w$ ,  $i = 1, 2$ . Hence  $\chi(G) \leq \chi(G[V_1]) + \chi(G[V_2]) \leq 2f_{\prec}(H - v)$ . If  $v$  is the leftmost or the rightmost in  $H$ , assume without loss of generality the former. Then clearly  $G - u \in \text{Forb}_{\prec}(H - v)$  for the leftmost vertex  $u$  of  $G$ . Thus  $\chi(G) \leq 1 + \chi(G - u) \leq 1 + f_{\prec}(H - v) \leq 2f_{\prec}(H - v)$ . Since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq 2f_{\prec}(H - v)$  in both cases.  $\square$

**Reduction Lemma 3.** *Let  $u$  and  $v$  be the leftmost and rightmost vertices in an ordered graph  $H$ ,  $|V(H)| \geq 4$ . If  $uv$  is an isolated edge in  $H$  and  $f_{\prec}(H - \{u, v\}) \neq \infty$ , then*

$$f_{\prec}(H) \leq 2f_{\prec}(H - \{u, v\}) + 1.$$

*Proof.* See Figure 4 (middle). Let  $H' = H - \{u, v\}$  and consider an ordered graph  $G \in \text{Forb}_{\prec}(H)$ . If  $G$  does not contain a copy of  $H'$ , then  $\chi(G) \leq f_{\prec}(H') \leq 2f_{\prec}(H') + 1$ . So, assume that  $G$  contains a copy of  $H'$ . Let  $V_1 \dot{\cup} \dots \dot{\cup} V_p$  denote a partition of  $V(G)$  into disjoint intervals with  $V_1 \prec \dots \prec V_p$ ,  $v_i$  being the leftmost vertex in  $V_i$ ,  $1 \leq i \leq p$ , such that  $G[V_i] \in \text{Forb}_{\prec}(H')$ ,  $1 \leq i \leq p$ , and  $G[V_i \cup \{v_{i+1}\}]$  contains a copy of  $H'$ ,  $1 \leq i < p$ . Note that one can find such a partition greedily by iteratively choosing a largest interval from the left that does not induce any copy of  $H'$  in  $G$ . If  $p \geq 3$ , there are no edges  $xy$  with  $x \in V_i$  and  $v_{i+2} \prec y$ , since otherwise  $xy$  together with a copy of  $H'$  in  $G[V_{i+1} \cup \{v_{i+2}\}]$  forms a copy of  $H$ ,  $1 \leq i \leq p - 2$ .

Choose a set  $\Phi$  of  $2f_{\prec}(H') + 1$  distinct colors. Let  $\Phi_1, \dots, \Phi_p \subset \Phi$  denote subsets of colors such that  $|\Phi_i| = f_{\prec}(H')$ ,  $1 \leq i \leq p$ ,  $\Phi_i \cap \Phi_{i+1} = \emptyset$ ,  $1 \leq i < p$ , and, if  $p \geq 3$ ,  $\Phi_{i+2} \setminus (\Phi_i \cup \Phi_{i+1}) \neq \emptyset$ ,  $1 \leq i \leq p - 2$ . Note that such sets  $\Phi_i$  can be chosen greedily from  $\Phi$ . Since  $G[V_i] \in \text{Forb}_{\prec}(H')$  we can color  $G[V_i]$  properly with colors from  $\Phi_i$ ,  $1 \leq i \leq p$ , such that, if  $i \geq 3$ ,  $v_i$  is colored with a color in  $\Phi_i \setminus (\Phi_{i-1} \cup \Phi_{i-2})$ . This yields a proper coloring of  $G$  using colors from the set  $\Phi$  only. Hence  $\chi(G) \leq 2f_{\prec}(H') + 1$ . Since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq 2f_{\prec}(H - \{u, v\}) + 1$ .  $\square$

Recall, that a vertex in an ordered graph  $H$  is called *reducible*, if it is a leaf in  $H$ , is leftmost or rightmost in  $H$  and has a common neighbor with the vertex next to it. See Figure 4 (right).

**Reduction Lemma 4.** *Let  $H$  denote an ordered graph with  $|V(H)| \geq 3$ . If  $u$  is a reducible vertex in  $H$  and  $f_{\prec}(H - u) \neq \infty$ , then*

$$f_{\prec}(H) \leq 2 f_{\prec}(H - u).$$

*Moreover, for each  $G \in \text{Forb}_{\prec}(H)$  there is  $G' \subseteq G$  such that  $G'$  is 1-degenerate and deleting the edges of  $G'$  from  $G$  yields a graph from  $\text{Forb}_{\prec}(H - u)$ .*

*Proof.* By reversing  $H$  if necessary we may assume that the reducible vertex  $u$  is leftmost in  $H$ . Let  $G \in \text{Forb}_{\prec}(H)$ . Let  $E$  denote the set of edges in  $G$  consisting for each vertex  $w$  in  $G$  of the longest edge to the left incident to  $w$  in  $G$ , if such an edge exists.

Assume that there is a copy  $H'$  of  $H - u$  in  $G - E$ . Let  $v$  denote the vertex in  $H'$  corresponding to the vertex immediately to the right of  $u$  in  $H$  and let  $w$  denote the vertex in  $H'$  corresponding to the neighbor of  $u$  in  $H$ . Then  $v$  is leftmost in  $H'$  and there is an edge between  $v$  and  $w$  in  $H'$ . Thus, there is an edge  $xw$  in  $E$  incident to  $w$  in  $G$  with  $x \prec v$ . Hence  $H'$  extends to a copy of  $H$  in  $G$  with the edge  $xw$ , a contradiction. This shows that  $G - E \in \text{Forb}_{\prec}(H - u)$ .

Finally observe that the graph  $G'$  with the edge-set  $E$  is 1-degenerate and hence 2-colorable. This shows that  $\chi(G) \leq \chi(G')\chi(G - E) \leq 2f_{\prec}(H - u)$  and since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq 2f_{\prec}(H - u)$ .  $\square$

Having Reduction Lemma 4 at hand, we are now ready to prove that every non-crossing monotonically alternating tree  $T$  satisfies  $f_{\prec}(T) \neq \infty$ .

**Lemma 4.4.** *If  $T$  is a non-crossing monotonically alternating tree with  $|V(T)| \geq 2$ , then*

$$f_{\prec}(T) \leq 2|V(T)| - 3.$$

*Proof.* Let  $k = |V(T)|$  and  $G \in \text{Forb}_{\prec}(T)$ . We shall prove that  $G$  can be edge-decomposed into  $(k - 2)$  1-degenerate graphs by induction on  $k$ .

If  $k = 2$ , then  $T$  consists of a single edge only. Hence  $G$  has an empty edge-set and there is nothing to prove.

So consider  $k \geq 3$  and assume that the induction statement holds for all smaller values of  $k$ . Assume for the sake of contradiction that the leftmost vertex  $u$  and the rightmost  $w$  in  $T$  are of degree at least 2. Then the longest and the shortest edge incident to  $w$  do not coincide. Let  $e$  be the longest edge incident to  $w$ . Since in a monotonically alternating tree each edge is the shortest edge incident to its left or right endpoint,  $e$  is the shortest edge incident to its left endpoint. In particular,  $e \neq uw$  because  $u$  is incident to another edge  $e'$ , shorter than  $uw$ . Thus  $e$  and  $e'$  cross since  $\chi_{\prec}(T) \leq 2$ , a contradiction. Hence the leftmost or the rightmost vertex is a leaf in  $T$ .

By reversing  $T$  if necessary we assume that  $u$  is of degree 1. We shall show that  $u$  is a reducible leaf. To do so, we need to show that the vertex  $x$  that is immediately to the right of  $u$  is adjacent to the neighbor  $v$  of  $u$ . Assume for the sake of contradiction that  $x$  is not adjacent to  $v$ . Note that  $v$  is adjacent to a leaf, so it

is not a leaf itself. Let  $e''$  be an edge incident to  $v$ ,  $e'' \neq uv$ . Then an edge incident to  $x$  crosses either  $uv$  or  $e''$  since  $\chi_{\prec}(T) \leq 2$ , a contradiction. Thus  $x$  is adjacent to  $v$  and  $u$  is a reducible leaf in  $T$ .

Therefore, by Reduction Lemma 4, there is a 1-degenerate subgraph  $G'$  of  $G$  such that removing the edges of  $G'$  from  $G$  yields a graph  $G'' \in \text{Forb}_{\prec}(T - u)$ . Observe that the tree  $T - u$  is non-crossing and monotonically alternating with  $k > |V(T - u)| = k - 1 \geq 2$ . Hence  $G''$  can be edge-decomposed into  $(k - 3)$  1-degenerate graphs  $G_1, \dots, G_{k-3}$  by induction. Thus the graphs  $G_1, \dots, G_{k-3}, G'$  decompose  $G$  into  $(k - 2)$  1-degenerate graphs, proving the induction step.

If  $k = 2$ , we know that  $G$  has no edges and  $\chi(G) = 1 \leq 2|V(T)| - 3$ . So assume that  $k \geq 3$ . Since  $G$  is a union of  $(k - 2)$  1-degenerate graphs, each subgraph of  $G$  is a union of  $(k - 2)$  1-degenerate graphs, so each subgraph  $G^*$  of  $G$  on at least one vertex that has at most  $(k - 2)(|V(G^*)| - 1)$  edges, and thus has a vertex of degree at most  $2(k - 2) - 1$ . Therefore  $G$  is  $(2(k - 2) - 1)$ -degenerate, so  $\chi(G) \leq 2(k - 2) \leq 2|V(T)| - 3$ . Since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq 2|V(T)| - 3$ .  $\square$

**Reduction Lemma 5.** *Let  $T$  denote an ordered matching on at least 2 edges. If  $uv$  is an edge in  $T$  and  $u$  and  $v$  are consecutive and  $f_{\prec}(T - \{u, v\}) \neq \infty$ , then*

$$f_{\prec}(T) \leq 3 f_{\prec}(T - \{u, v\}).$$

*Proof.* Let  $G \in \text{Forb}_{\prec}(T)$  with vertices  $v_1 \prec \dots \prec v_n$ . We shall prove that  $\chi(G) \leq 3 f_{\prec}(T - \{u, v\})$  by induction on  $n = |V(G)|$ . If  $n \leq 3 f_{\prec}(T - \{u, v\})$ , then the claim holds trivially. So assume that  $n > 3 f_{\prec}(T - \{u, v\}) \geq 3$ . If there are two consecutive vertices  $x, y$  in  $G$  that are not adjacent, then let  $G'$  denote the graph obtained by identifying  $x$  and  $y$ . Then  $G' \in \text{Forb}_{\prec}(T)$  and  $\chi(G) \leq \chi(G')$ . Hence  $\chi(G) \leq \chi(G') \leq 3 f_{\prec}(T - \{u, v\})$  by induction. If each pair of consecutive vertices in  $G$  forms an edge, then consider a partition  $V(G) = V_0 \dot{\cup} V_1 \dot{\cup} V_2$  such that  $V_i = \{v_j \in V(G) \mid j \equiv i \pmod{3}\}$ . Observe that for each pair of vertices  $x, y \in V_i$  there are at least two adjacent vertices from  $V(G) \setminus V_i$  between  $x$  and  $y$ . Hence  $G[V_i] \in \text{Forb}_{\prec}(T - \{u, v\})$ ,  $i = 0, 1, 2$ , since any copy of  $T - \{u, v\}$  in  $G[V_i]$  extends to a copy of  $T$  in  $G$ . Hence  $\chi(G) \leq 3 f_{\prec}(T - \{u, v\})$  and since  $G \in \text{Forb}_{\prec}(H)$  was arbitrary we have  $f_{\prec}(H) \leq 3 f_{\prec}(T - \{u, v\})$ .  $\square$

## 5 Proofs of Theorems

### 5.1 Proof of Theorem 1

We will prove that if an ordered graph  $H$  contains a cycle, a tangled path or a bonnet then for each positive integer  $k$  there is an ordered graph  $G \in \text{Forb}_{\prec}(H)$  with  $\chi(G) \geq k$ .

First assume that  $H$  contain a cycle of length  $\ell$ . Fix a positive integer  $k$  and consider a graph  $G$  of girth at least  $\ell + 1$  and chromatic number at least  $k$  that exists



Figure 5: A graph  $G_k$  obtained by Tutte's construction from a graph  $G_{k-1}$ . Here  $G_k[U_i] = G_{k-1}$ ,  $1 \leq i \leq M$ .

by [11]. Then no ordering of the vertices of  $G$  gives an ordered subgraph isomorphic to  $H$ . This shows that for any positive integer  $k$ ,  $f_{\prec}(H) \geq k$  and hence  $f_{\prec}(H) = \infty$ .

A tangled path is minimal if it does not contain a proper subpath that is tangled. Next we shall show that for each minimal tangled path  $P$  and each  $k \geq 1$  there is an ordered graph  $G_k \in \text{Forb}_{\prec}(P)$  with  $\chi(G_k) \geq k$ .

By reversing  $P$  if necessary we assume that in  $P$  the paths  $Pu$  and  $uP$  cross for the rightmost vertex  $u$  in  $P$ . We will prove the claim by induction on  $k$ . If  $k \leq 3$  let  $G_k = K_k$  that has no crossing edges and thus no tangled paths. Consider  $k \geq 4$  and let  $G_{k-1}$  denote an  $n$ -vertex graph of chromatic number at least  $k-1$  that does not contain a copy of  $P$ . Such a graph exists by induction. The following construction is due to Tutte (alias Blanche Descartes) for unordered graphs [9]. Let  $N = (k-1)(n-1) + 1$  and  $M = \binom{N}{n}$ . Consider pairwise disjoint sets of vertices  $U_1, \dots, U_M, V$  such that  $|U_i| = n$ ,  $i = 1, \dots, M$ ,  $|V| = N$  and  $U_1 \prec \dots \prec U_M \prec V$ . Let  $V_1, \dots, V_M$  be the  $n$ -element subsets of  $V$ . Let each  $U_i$ ,  $i = 1, \dots, M$ , induce a copy of  $G_{k-1}$ . Finally let there be a perfect matching between  $U_i$  and  $V_i$  such that the  $j^{\text{th}}$  vertex in  $U_i$  is matched to the  $j^{\text{th}}$  vertex in  $V_i$ ,  $i = 1, \dots, M$ . See Figure 5.

First we shall show that  $\chi(G_k) \geq k$ . If there are at most  $k-1$  colors assigned to the vertices of  $G_k$ , then by Pigeonhole Principle there are  $n$  vertices of  $V$  of the same color, i.e., there is a set  $V_i$  with all vertices of the same color, say color 1. Since each vertex of  $U_i$  is adjacent to a vertex in  $V_i$ , no vertex in  $U_i$  is colored 1, so if the coloring is proper, then  $G[U_i]$  uses at most  $k-2$  colors. Hence the coloring is not proper, since  $\chi(G[U_i]) = \chi(G_{k-1}) \geq k-1$ . Therefore  $\chi(G_k) \geq k$ .

Now, we shall show that  $G_k$  does not contain a copy of  $P$ . Assume that there is such a copy  $P'$  of  $P$  in  $G_k$  with rightmost vertex  $u$  of  $P'$ . Let  $x$  and  $y$  be the neighbors of  $u$  in  $P'$ , i.e.,  $P'$  is a union of paths  $P'yu$  and  $uxP'$ . Then  $u \in V$  and  $x, y \notin V$ , since  $G[U_i]$  does not contain a copy of  $P$  and there are no edges in  $G_k[V]$ . Let  $x \in U_i$  and  $y \in U_j$ . Note that  $i \neq j$  because the edges between  $U_i$  and  $V$  form a matching. The path  $uxP'$  is a proper subpath of  $P'$  and hence is not tangled. Recall that for each edge  $zw$  with  $z \in U_i$ ,  $w \in V$ , and  $w \prec u$ , we have  $z \prec x$  due to the construction of the matching between  $U_i$  and  $V_i$ . Hence the path  $uxP'$  does not contain any vertex  $w \in V$  with  $w \prec u$ , since otherwise the path  $uxP'w$  has a vertex left of  $x$  contradicting Lemma 4.1 applied to  $u$ ,  $x$  and  $w$ . Hence  $V(xP') \subseteq U_i$ , because there are no edges between  $U_i$ 's and  $u$  is rightmost in  $P'$ . See Figure 6. Similarly, all vertices of  $P'y$  are contained in  $U_j$ . Thus  $P'u$  and  $uP'$  do not cross. However,  $P'$  is a copy of  $P$  with respective subpaths crossing, a contradiction. Hence  $G_k \in \text{Forb}_{\prec}(P)$ .



Figure 6: A path in  $G_k$  with rightmost vertex  $u \in V$  is not tangled if  $Pu$  and  $uP$  are not tangled.

Now, if an ordered graph  $H$  contains a tangled path, then it contains a minimal tangled path. Thus  $f_{\prec}(H) = \infty$ .

Now, let  $B$  be a bonnet. By reversing  $B$  if necessary, we assume that  $B$  has vertices  $u \prec v \preceq x, y \preceq w$  and edges  $uv, uw, xy$ . A shift graph  $S(n)$  is defined on vertices  $\{(i, j) \mid 1 \leq i < j \leq n\}$  and edges  $\{(i, j), (j, t)\} \mid 1 \leq i < j < t \leq n\}$ . We will show that some ordering of  $S(n)$  does not contain  $B$ . Let  $G = S(n)$  be a shift graph with vertices ordered lexicographically, i.e.,  $(x_1, x_2) \prec (y_1, y_2)$  if and only if  $x_1 < y_1$ , or  $x_1 = y_1$  and  $x_2 < y_2$ . Assume that  $G$  contains vertices  $u = (u_1, u_2)$ ,  $v = (v_1, v_2)$ ,  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$  and  $w = (w_1, w_2)$  that form a copy of  $B$  with  $u \prec v \preceq x, y \preceq w$  and edges  $uv, uw, xy$ . Then  $u_2 = v_1$ ,  $u_2 = w_1$ ,  $x_2 = y_1$ . Thus  $v_1 = w_1$ . However, since  $v \preceq x, y \preceq w$ , we have that  $v_1 \leq x_1, y_1 \leq w_1$ , so  $x_1 = y_1 = v_1 = w_1$ . But  $x_2 = y_1$ , thus  $x_2 = x_1$ , a contradiction. Thus  $G \in \text{Forb}_{\prec}(B)$ . We claim that  $\chi(G) \geq \log(n) \geq \log c|V(G)|$ . Indeed consider a proper coloring  $\phi$  of  $G$  using  $\chi(G)$  colors and sets of colors  $\Phi_i = \{\phi(i, j) \mid i < j \leq n\}$ ,  $1 \leq i \leq n$ . Then  $\phi(i, j) \notin \Phi_j$ , since a vertex  $(i, j)$  is adjacent to all vertices  $(j, t)$ ,  $j < t \leq n$ . Therefore  $\Phi_i \neq \Phi_j$  for all  $j < i$ . Hence all the sets of colors are distinct. This shows that  $2^{\chi(G)} \geq n$ , since there are at most  $2^{\chi(G)}$  distinct subsets of colors. This proves that  $\chi(G) \geq \log(n)$ . Thus, for any  $k$ , there is an ordered graph of chromatic number at least  $k$  in  $\text{Forb}_{\prec}(B)$ . So, if an ordered graph  $H$  contains a bonnet, then  $f_{\prec}(H) = \infty$ .  $\square$

## 5.2 Proof of Theorem 2

Let  $T'$  be a segment of an ordered tree that does not contain a bonnet or a tangled path. We shall prove that  $T'$  is monotonically alternating by induction on  $k = |V(T')|$ . Every ordered tree on at most two vertices is monotonically alternating. So suppose  $k \geq 3$ . We have  $\chi_{\prec}(T') = 2$  due to Lemma 4.3.

**Claim.** *The leftmost or the rightmost vertex in  $T'$  is of degree 1.*

*Proof of Claim.* For the sake of contradiction assume that both the leftmost vertex  $u$  and the rightmost vertex  $v$  in  $T'$  are of degree at least 2. If  $u$  and  $v$  are adjacent then the edge  $uv$ , another edge incident to  $u$  and another edge incident to  $v$  form a tangled path since  $\chi_{\prec}(T') = 2$ , a contradiction. If  $u$  and  $v$  are not adjacent let  $P$  denote the path in  $T'$  connecting  $u$  and  $v$ . It uses at most one of the edges incident to  $u$ . Then any other edge  $zu$  incident to  $u$  crosses the edge in  $P$  that is incident to  $v$  since  $\chi_{\prec}(T') = 2$ . Hence  $zP$  forms a tangled path, a contradiction. This shows that at least one of  $u$  or  $v$  is a leaf in  $T'$ .  $\triangle$

By reversing  $T'$  if necessary we assume that the leftmost vertex  $u$  is a leaf in  $T'$ . The ordered tree  $T' - u$  is monotonically alternating by induction and Lemma 4.2. Consider the partition  $V(T') = L \dot{\cup} R$ , with  $L \prec R$  and  $L$  and  $R$  being independent sets. Such a partition is unique since  $T'$  is connected. Let  $v$  be the neighbor of  $u$  in  $T'$ . Since  $\chi_{\prec}(T') = 2$ ,  $v \in R$ . Since  $T'$  is connected,  $k \geq 3$  and  $u$  is leftmost in  $T'$ , the edge  $uv$  is not the shortest edge incident to  $v$ . Hence  $uv \notin S(R)$  and therefore  $S(R)$  has no crossing edges by induction. Clearly  $uv \in S(L)$  since  $uv$  is the only edge incident to  $u$  and thus it is the shortest incident to  $u$  edge. If  $uv$  crosses some edge  $xy$  in  $T'$ ,  $x \prec y$ , then all vertices in the path connecting  $v$  and  $x$  are between  $x$  and  $v$  due to Lemma 4.1 applied to  $x$ ,  $y$  and  $v$ . Therefore  $xy$  is not the shortest edge incident to  $x$  and hence  $xy \notin S(L)$ . This shows that  $S(L)$  has no crossing edges and thus  $T'$  is monotonically alternating.

The other way round assume that each segment of an ordered tree  $T$  is monotonically alternating. We need to show that each segment contains neither a bonnet nor a tangled path. Let  $T'$  denote a segment of  $T$ ,  $V(T') = L \cup R$ ,  $L \prec R$  and  $E(T') = S(L) \cup S(R)$ , so each edges is either a shortest edge incident to a vertex in  $R$  or a shortest edge incident to a vertex in  $L$ . Then  $\chi_{\prec}(T') \leq 2$  and hence  $T'$  does not contain a bonnet. We will prove that  $T'$  does not contain a tangled path by induction on  $k = |V(T')|$ . If  $k \leq 3$ , then there are no crossing edges in  $T'$  and hence no tangled path. Suppose  $k \geq 4$ .

Assume that the leftmost vertex  $u$  and the rightmost vertex  $w$  in  $T'$  are of degree at least 2. If  $uw \in E(T')$  then  $uw \notin S(L)$  and  $uw \notin S(R)$ , a contradiction. So,  $uw \notin E(T')$ . Consider the longest edge  $xw$  incident to  $w$ . Then  $x \neq u$  and since  $xw \notin S(R)$ ,  $xw \in S(L)$ . Then the shortest edge incident to  $u$  crosses  $xw$ , a contradiction since  $S(L)$  does not contain crossing edges. Hence the leftmost or the rightmost vertex is a leaf in  $T'$ .

By reversing  $T'$  if necessary we assume that the leftmost vertex  $u$  is a leaf. We see that  $T' - u$  is monotonically alternating, thus by induction it does not contain a tangled path. Hence if  $T'$  has a tangled path  $P$ , then  $P$  contains an edge  $uv$  crossing some other edge in  $P$ , where  $v$  is the neighbor of  $u$  in  $T'$ . Then the rightmost vertex  $r$  in  $P$  is of degree 2 and to the right of  $v$ , since  $P$  is tangled and  $u$  is leftmost and of degree 1 in  $T'$ . Let  $x$  and  $y$ ,  $x \prec y$ , be neighbors of  $r$  in  $P$ . Then  $xr$  is the shortest edge incident to  $x$ , since any shorter edge forms a tangled path with  $r$  and  $y$  in  $T' - u$ . This is a contradiction since  $uv$  and  $xr$  cross and  $T'$  is monotonically alternating. Thus  $T'$  has no tangled path.

Finally we prove the last statement of the theorem. If  $H$  is a connected ordered graph with  $f_{\prec}(H) \neq \infty$ , then  $H$  is a tree that contains neither a bonnet nor a tangled path due to Theorem 1. Hence each segment of  $H$  is a monotonically alternating tree.  $\square$

### 5.3 Proof of Theorem 3

Let  $T$  be a non-crossing ordered graph such that  $f_{\prec}(T) \neq \infty$ . Then  $T$  is acyclic, contains no tangled path and no bonnet by Theorem 1. Hence  $T$  is a non-crossing ordered forest with no bonnet.

On the other hand let  $T$  be a non-crossing forest with no bonnet. Recall that  $f_{\prec}(H) \geq k - 1$  for each ordered  $k$ -vertex graph  $H$  because  $K_{k-1} \in \text{Forb}_{\prec}(H)$ . We shall prove that  $f_{\prec}(T) \neq \infty$ . Let  $k = |V(T)|$  and consider any ordered graph  $G \in \text{Forb}_{\prec}(T)$ . We will prove by induction on  $k$  that  $\chi(G) \leq 2^k$  and  $\chi(G) \leq 2k - 3$  if  $T$  is a tree. If  $k = 2$ , then clearly  $\chi(G) = 1$ . So consider  $k \geq 3$ .

If  $T$  is a tree, then each segment of  $T$  is a monotonically alternating tree, by Theorem 2. If there is only one segment in  $T$ , then  $f_{\prec}(T) \leq 2k - 3$  by Lemma 4.4. If there is more than one segment in  $T$ , then there is an inner cut vertex splitting  $T$  into two trees  $T_1$  and  $T_2$  that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have  $f_{\prec}(T) \leq f_{\prec}(T_1) + f_{\prec}(T_2) \leq 2|V(T_1)| - 3 + 2|V(T_2)| - 3 = 2(|V(T)| + 1) - 6 = 2k - 4$ .

If  $T$  is a forest we consider several cases. If  $T$  has more than one segment, then there is an inner cut vertex splitting  $T$  into two forests  $T_1$  and  $T_2$  that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 1 and induction we have  $f_{\prec}(T) \leq f_{\prec}(T_1) + f_{\prec}(T_2) \leq 2^{|V(T_1)|} + 2^{|V(T_2)|} = 2^t + 2^{k+1-t} \leq 2^k$  with  $t = |V(T_1)| \geq 2$ . If  $T$  has an isolated vertex  $u$ , then by Reduction Lemma 2 and induction we have  $f_{\prec}(T) \leq 2f_{\prec}(T - u) \leq 2 \cdot 2^{k-1} = 2^k$ . Finally, if  $T$  has no isolated vertices and exactly one segment, then consider the leftmost and rightmost vertices  $u$  and  $v$  of  $T$ . Since  $u$  and  $v$  are not isolated in this case, and  $T$  is non-crossing with no inner cut vertices,  $uv$  is an edge. If  $uv$  is isolated, then  $k \geq 4$  (since there is no isolated vertex) and by Reduction Lemma 3 and induction we have  $f_{\prec}(T) \leq 2 \cdot f_{\prec}(T - \{u, v\}) + 1 \leq 2 \cdot 2^{k-2} + 1 \leq 2^k$ . If  $uv$  is not isolated, then either  $u$  or  $v$ , say  $u$ , is a leaf of  $T$ , since  $T$  is non-crossing and does not contain a bonnet. Let  $xv$  denote the longest edge incident to  $v$  in  $T - u$ . Note that  $x$  exists since the edge  $uv$  is not isolated. Then there is no other vertex between  $u$  and  $x$ , since such a vertex would be isolated in the non-crossing forest  $T$  without bonnets. Thus,  $u$  is a reducible vertex, so by Reduction Lemma 4 and induction we have  $f_{\prec}(T) \leq 2f_{\prec}(T - u) \leq 2 \cdot 2^{k-1} = 2^k$ .

Next, we provide a  $k$ -vertex non-crossing tree with no bonnet such that  $\infty \neq f_{\prec}(T) \geq k$ . Let  $T$  be a monotonically alternating path on  $k \geq 4$  vertices with leftmost vertex of degree 1, as in Figure 7 (right). Further let  $G$  denote a graph on vertices  $u \prec x_1 \prec \dots \prec x_{k-2} \prec y_1 \prec \dots \prec y_{k-2} \prec x \prec y$  such that  $xy$  is an edge and  $\{u, x_1, \dots, x_{k-2}\}$ ,  $\{u, y_1, \dots, y_{k-2}\}$ ,  $\{x, x_1, \dots, x_{k-2}\}$ , and  $\{y, y_1, \dots, y_{k-2}\}$  induce complete graphs on  $k - 1$  vertices each. See Figure 7 (left).

We shall show that  $G \in \text{Forb}_{\prec}(T)$  and  $\chi(G) \geq k$ . Consider a proper vertex

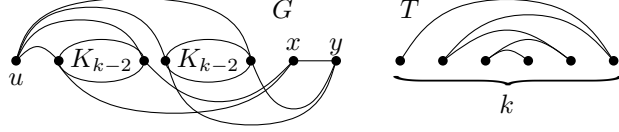


Figure 7: An ordered graph  $G$  with chromatic number  $k$  not containing a non-crossing and ordered tree  $T$  on  $k$  vertices without bonnets on the right,  $k = 6$ .

coloring of  $G$  using colors  $1, \dots, k-1$ . Without loss of generality  $u$  has color 1. Then all colors  $2, \dots, k-1$  are used on the vertices  $x_1, \dots, x_{k-2}$  as well as on  $y_1, \dots, y_{k-2}$ . Hence both  $x$  and  $y$  are of color 1, a contradiction. Thus  $\chi(G) \geq k$ .

Assume that there is a copy  $P$  of  $T$  in  $G$ . Let  $v$  be the leftmost and  $w$  be the rightmost vertex in  $P$ . Note that  $vw$  is an edge and that there are  $k$  vertices between  $v$  and  $w$ . Therefore  $vw$  is one of the edges  $uy_i$ ,  $1 \leq i \leq k-2$ ,  $x_jx$ ,  $1 \leq j \leq k-2$ , or  $y_1y$ . In the first case  $V(P) \subseteq \{u, y_1, \dots, y_{k-2}\}$ , in the second case  $V(P) \subseteq \{x_1, \dots, x_{k-2}, x\}$  and in the last case either  $P = y_1, y, x$  or  $V(P) \subseteq \{y, y_1, \dots, y_{k-2}\}$ . Since  $T$  has at least 4 vertices,  $P \neq y_1, y, x$ . So in any case  $P$  has at most  $k-1$  vertices, a contradiction since  $T$  has  $k$  vertices. Hence  $G \in \text{Forb}_{\prec}(T)$ .

Finally it is easy to see that  $f_{\prec}(T) = k-1$  for any ordered tree  $T$  on at most 3 vertices using Reduction Lemmas 1 and 4.  $\square$

#### 5.4 Proof of Theorem 4

- Let  $T$  be an ordered forest on  $k$  vertices where each segment is a generalized star, a 2-nesting, or a 2-crossing. Let  $T_1, \dots, T_s$  denote the segments of  $T$  and  $k_i = |V(T_i)|$ ,  $1 \leq i \leq s$ . Let  $T'$  be a segment of  $T$ . If  $T'$  is a generalized star on  $k'$  vertices, then the center of the star is leftmost (or rightmost) in  $T'$ . Let  $G \in \text{Forb}_{\prec}(T')$ . Then each vertex in  $G$  has at most  $k' - 2$  neighbors to the right (or to the left). Thus each such graph can be greedily colored from right to left (or left to right) with at most  $k' - 1$  colors. This shows that  $f_{\prec}(T') \leq |V(T')| - 1$ . If  $T'$  is a 2-nesting, then  $f_{\prec}(T') = 3 = |V(T')| - 1$  due to [10] (Lemma 9). If  $T'$  is a 2-crossing, then  $f_{\prec}(T') = 3 = |V(T')| - 1$ , since any graph not containing  $T'$  is outerplanar and outerplanar graphs have chromatic number at most 3. We apply Reduction Lemma 1 and the results above which yield  $f_{\prec}(T) \leq \sum_{i=1}^s f_{\prec}(T_i) \leq \sum_{i=1}^s (k_i - 1) = k - 1$ .
- Let  $T$  be an ordered forest on  $k$  vertices where each segment is a generalized star, a non-crossing tree without bonnets, a crossing or a nesting. Let  $T_1, \dots, T_s$  denote the segments of  $T$  and  $k_i = |V(T_i)| \geq 2$ . Let  $T'$  be a segment of  $T$ . If  $T'$  is a  $k'$ -nesting or a  $k'$ -crossing,  $k' \geq 2$ , then  $f_{\prec}(T') \leq 4(k' - 1) \leq 2|V(T')| - 3$  due to equation (1), since any graph  $G \in \text{Forb}_{\prec}(T')$  contains less than  $2(k' - 1)|V(G)|$  edges due to Dujmovic and Wood [10] (for nestings), respectively Capovileas and Pach [5] (for crossings). Further  $f_{\prec}(T') \leq 2|V(T')| - 3$  if  $T'$  is a non-crossing tree without bonnets



due to Theorem 3. Hence Reduction Lemma 1 yields  $f_{\prec}(T) \leq \sum_{i=1}^s f_{\prec}(T_i) \leq \sum_{i=1}^s (2k_i - 3) \leq 2k - 3$ .

- Let  $T = M(t, m, \pi)$  for some positive integers  $m$  and  $t$  and a permutation  $\pi$  of  $[t]$ . If  $t = 1$ , then  $f_{\prec}(T) = m$  due to the results above, since  $M(1, m, \pi)$  is a star on  $m + 1$  vertices. Weidert [19] proves that  $\text{ex}_{\prec}(n, M(t, 1, \pi)) \leq \text{ex}_{\prec}(n, M(t, 2, \pi)) \leq 11t^4 \binom{2t^2}{2t} n < t^4 (2t^2)^{2t} n$  for any positive integer  $t \geq 2$  and any permutation  $\pi$  of  $[t]$ . Moreover if  $m \geq 2$ , then

$$\text{ex}_{\prec}(n, M(t, m, \pi)) \leq 2^{t(m-2)} \text{ex}_{\prec}(n, M(t, 2, \pi))$$

due to a reduction by Tardos [18]. Therefore  $\text{ex}_{\prec}(n, M(t, m, \pi)) < 2^{tm} t^{4+4t} n$ . Thus, using the fact that  $|V(T)| = k = tm + t$  and equation (1) we have that  $f_{\prec}(M(t, m, \pi)) \leq 2^{tm+9t \log(t)} \leq 2^{10k \log k}$ .

- Conlon *et al.* [7] and independently Balko *et al.* [2] prove that there is a positive constant  $c$  such that for any sufficiently large positive integer  $k$  there is an ordered matchings on  $k$  vertices with ordered Ramsey number at least  $2^{c \frac{\log(k)^2}{\log \log(k)}}$ . If, for some ordered graph  $H$ , the edges of a complete ordered graph  $G$  on  $N = R_{\prec}(H) - 1$  vertices are colored in two colors without monochromatic copies of  $H$ , then both color classes form ordered graphs  $G_1$  and  $G_2$  in  $\text{Forb}_{\prec}(H)$ . Then one of the  $G_i$ 's has chromatic number at least  $\sqrt{N}$ , since a product of proper colorings of  $G_1$  and  $G_2$  yields a proper coloring of  $G$  using  $\chi(G_1)\chi(G_2) \geq \chi(G) = N$  colors. This shows that there is a positive constant  $c'$  such that for all positive integers  $k$  and ordered matchings  $H$  on  $k$  vertices with  $f_{\prec}(H) \geq 2^{c' \frac{\log(k)^2}{\log \log(k)}}$ .  $\square$

## 6 Small Forests

Let  $P_k$  denote a path on  $k$  vertices,  $M_k$  a matching on  $k$  edges and  $S_k$  a star with  $k$  leaves (note that  $M_1 = S_1 = P_2$  and  $P_3 = S_2$ ). Further let  $G + H$  denote the vertex disjoint union of graphs  $G$  and  $H$ . Then the set of all forests without isolated vertices and at most 3 edges is given by

$$\{P_2, S_2, M_2, S_3, P_4, S_2 + P_2, M_3\}.$$

Let  $G$  denote a graph on  $n$  vertices and  $a$  automorphisms. Then the number  $\text{ord}(G)$  of non-isomorphic orderings of  $G$  equals  $\text{ord}(G) = \frac{n!}{a}$ . Hence

$$\begin{aligned} \text{ord}(P_2) &= \frac{2!}{2} = 1, \quad \text{ord}(S_2) = \frac{3!}{2} = 3, \quad \text{ord}(M_2) = \frac{4!}{8} = 3, \quad \text{ord}(S_3) = \frac{4!}{3!} = 4, \\ \text{ord}(P_4) &= \frac{4!}{2} = 12, \quad \text{ord}(S_2 + P_2) = \frac{5!}{2 \cdot 2} = 30, \quad \text{ord}(M_3) = \frac{6!}{6 \cdot 4 \cdot 2} = 15. \end{aligned}$$

Recall that the reverse  $\bar{T}$  of an ordered graph  $T$  is the ordered graph obtained by reversing the ordering of the vertices in  $T$ . Note that  $f_{\prec}(T) = f_{\prec}(\bar{T})$  for any

ordered graph  $T$  since  $G \in \text{Forb}_{\prec}(T)$  if and only if  $\overline{G} \in \text{Forb}_{\prec}(\overline{T})$ . Table 8 shows all ordered forests  $T$  without isolated vertices and at most 3 edges and their  $f_{\prec}$  values, where only one of  $T$  and  $\overline{T}$  is listed. So when  $T$  and  $\overline{T}$  are not isomorphic ordered graphs the entry in the table represents two graphs. Such cases are marked with an  $*$ . For example there are only two instead of three entries for  $S_2$  and similarly for the other graphs.

## 7 Conclusions

In this paper, we consider the function  $f_{\prec}(H) = \sup\{\chi(G) \mid G \in \text{Forb}_{\prec}(H)\}$  for ordered graphs  $H$  on at least 2 vertices. We prove that in contrast to unordered and directed graphs,  $f_{\prec}(H) = \infty$  for some ordered forests  $H$ . To this end we explicitly describe several infinite classes of minimal ordered forests  $H$  with  $f_{\prec}(H) = \infty$ . A full answer to the following question remains open.

**Question 1.** *For which ordered forests  $H$  does  $f_{\prec}(H) = \infty$  hold?*

We completely answer Question 1 for non-crossing ordered graphs  $H$ . Suppose that  $H$  is a non-crossing ordered  $k$ -vertex graph with  $f_{\prec}(H) \neq \infty$ . We prove that, if  $H$  connected, then  $k - 1 \leq f_{\prec}(H) \leq 2k - 3$  and, if  $H$  is disconnected, then  $k - 1 \leq f_{\prec}(H) \leq 2^k$ . In addition, we give infinite classes of graphs for which  $f_{\prec}(H) = |V(H)| - 1$ , as well as infinite classes of graphs for which  $|V(H)| \leq f_{\prec}(H) \neq \infty$ . Note that we do not know whether  $f_{\prec}(H) \neq \infty$  for the matchings in the last statement of Theorem 4. For crossing connected ordered graphs, we reduce Question 1 to monotonically alternating trees:

**Question 2.** *For which monotonically alternating trees  $H$  does  $f_{\prec}(H) = \infty$  hold?*

We do not have an answer to Question 2 even for some monotonically alternating paths. A smallest unknown such path is  $u_5 u_1 u_3 u_2 u_4$ , where  $u_1 \prec \dots \prec u_5$ . See Figure 9 (left). The situation becomes even more unclear for crossing disconnected graphs. We do not know the value of  $f_{\prec}(H)$  for some ordered matchings  $H$ . A smallest such matching has edges  $u_1 u_3$ ,  $u_2 u_5$  and  $u_4 u_6$  where  $u_1 \prec \dots \prec u_6$ . See Figure 9 (right). Note that Reduction Lemmas 1, 2, 3 and 4 apply to crossing ordered graph as well. We find a more precise version of Reduction Lemma 2 and other types of reductions, similar to reductions for matrices in [18], but none of these lead to significantly better upper bounds in Theorems 3 and 4 or a new class of forests with finite  $f_{\prec}$ . The following question remains open, even when restricted to non-crossing graphs.

**Question 3.** *For  $k \geq 4$ , what is the value of the function*

$$f_{\prec}(k) = \max\{f_{\prec}(H) \mid |V(H)| = k, f_{\prec}(H) \neq \infty\}?$$

$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	1 (Thm. 4)	2 * (Thm. 4)	2 (Thm. 4)	3 (Thm. 4)	3 (Thm. 4)	3 (Thm. 4)
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	3 * (Thm. 4)	3 * (Thm. 4)	3 (Thm. 4)	$\infty$ * (bonnet)	$\infty$ * (tangled)	
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	3 * (Thm. 4)	$\infty$ (bonnet)	$\infty$ (tangled)	4 * (Lem. 4.4, Fig. 7)	$\leq 4$ (Red. 4)	
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	4 * (Thm. 4)	$\leq 6$ * (Red. 4)	? * (Thm. 4)	$\leq 6$ * (Red. 3)	$\leq 6$ * (Lem. 4.4)	
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	? * (Thm. 4)	$\neq \infty$ * (Thm. 4)	$\infty$ * (bonnet)	? * (Thm. 4)	4 * (Thm. 4)	
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	4 * (Thm. 4)	4 * (Thm. 4)	? * (Thm. 4)	$\leq 6$ (Red. 3)	4 * (Thm. 4)	?
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	5 (Thm. 4)	5 * (Thm. 4)	5 * (Thm. 4)	$\leq 9$ * (Red. 5)	$\leq 7$ (Red. 3)	
$\mathbf{T}$						
$\mathbf{f}_{\prec}(\mathbf{T})$	? (Thm. 4)	$\neq \infty$ * (Thm. 4)	$\leq 9$ (Red. 5)	$\leq 8$ (Thm. 4)	$\leq 7$ (Red. 3)	$\leq 8$ (Thm. 4)

Figure 8: All ordered forests  $T$  on at most 3 edges without isolated vertices and their  $f_{\prec}$  value.

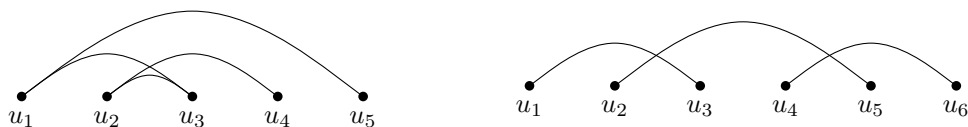


Figure 9: Ordered graphs  $H$  for which we don't know whether  $f_{\prec}(H) = \infty$ .

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